

# A non-archimedean analogue of Calabi-Yau theorem for totally degenerate abelian varieties

Yifeng Liu  
Columbia University

June 11, 2010

## Abstract

We show an example of a non-archimedean version of the Calabi-Yau theorem in complex geometry. Precisely, we consider totally degenerate abelian varieties and certain probability measures on their associated analytic spaces in the sense of Berkovich.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Mumford's construction</b>	<b>3</b>
<b>3</b>	<b>Toric metrized line bundles</b>	<b>4</b>
<b>4</b>	<b>A limit formula for the measure</b>	<b>10</b>
<b>5</b>	<b>A Calabi-Yau theorem</b>	<b>16</b>
	<b>References</b>	<b>18</b>

## 1 Introduction

The theorem of Calabi-Yau is one of most important results in complex geometry which has many applications (e.g. Yau's famous paper [14]). In one version, it claims the following fact. Let  $M$  be a compact complex manifold with an ample line bundle  $L$ . Then for any smooth positive measure  $\mu$  on  $M$  with  $\int_M \mu = \int_M c_1(L)^{\dim M}$ , there is a positive metric  $\|\cdot\|$  on  $L$ , unique up to a constant multiple, such that  $c_1(L, \|\cdot\|)^{\dim M} = \mu$ .

One would like to ask the similar question for a non-archimedean field, for example, if we replace  $\mathbb{C}$  by  $\mathbb{C}_p$  and complex manifolds by non-archimedean analytic spaces, or Berkovich spaces in [1]. Hence let  $X$  be a smooth proper (strictly) analytic space over  $\mathbb{C}_p$  with an ample line bundle  $L$  (in particular, it is algebrizable by GAGA). Given any integrable metric  $\|\cdot\|$  (cf. [18]) on  $L$ , although we don't have a nice analogue of  $(1,1)$ -form for  $c_1(L, \|\cdot\|)$  in the non-archimedean situation so far, we can still talk about its top wedge, i.e.,  $c_1(L, \|\cdot\|)^{\dim M}$  which is defined by Chambert-Loir in [5]. The top wedge is a measure on the underlying compact (metrizable) topological space of  $X$ . If  $\|\cdot\|$  is semi-positive in the sense of [17, 18], then  $c_1(L, \|\cdot\|)^{\dim M}$  is a positive measure in the following sense:  $\int_X f c_1(L, \|\cdot\|)^{\dim M} \geq 0$  for any non-negative

continuous real function  $f$  on  $X$ . Then analogous to the complex case, given a positive measure  $\mu$  on  $X$  with  $\int_X \mu = \deg_L(X)$ , we can ask the following two questions:

(E) Does it exist a semi-positive metric  $\| \cdot \|$  on  $L$  such that  $c_1(L, \| \cdot \|)^{\dim M} = \mu$ ?

(U) If it does, is it unique up to a constant multiple?

The question (U) has been perfectly answered by Yuan and Zhang in [16]. There, they have already used this uniqueness result to prove several exciting theorems in algebraic dynamic systems. The answer to the question (E) is in fact *negative* in general which is due to several reasons. We are still not clear about the nature of this question. One possible reason is that the notion of being positive in the metric side and measure side are not quite compatible as in the complex case. Nevertheless, we would like to give one example where the answer to (E) is positive, of course, under certain restrictions.

The following is our situation. Let us consider a totally degenerate abelian variety  $A$ , say over  $\mathbb{C}_p$ , i.e., the associated analytic space  $A^{\text{an}} \cong (\mathbb{G}_m^d)^{\text{an}}/M$  for a complete lattice  $M \in \mathbb{G}_m^d(\mathbb{C}_p)$  (cf. Section 2). We have an evaluation map  $\tau_A : A^{\text{an}} = (\mathbb{G}_m^d)^{\text{an}}/M \rightarrow \mathbb{R}^d/\Lambda$  with a complete lattice  $\Lambda \subset \mathbb{R}^d$ , which is continuous and surjective. This map has a continuous section  $i_A : \mathbb{R}^d/\Lambda \hookrightarrow A^{\text{an}}$ . In fact,  $i_A$  identifies  $\mathbb{R}^d/\Lambda$  as a strong deformation retract of  $A^{\text{an}}$  for which  $i_A \circ \tau_A = \Phi(\cdot, 1)$  for a strong retraction map  $\Phi : A^{\text{an}} \times [0, 1] \rightarrow A^{\text{an}}$ . Then we prove the following theorem which is a certain non-archimedean analogue of Calabi-Yau theorem.

**Theorem** (Theorem 5.2). *Let  $A$  be a  $d$ -dimensional totally degenerate abelian variety over  $k$  (e.g.  $\mathbb{C}_p$ ) and  $L$  an ample line bundle on  $A$ . For any measure  $\mu = f d\mathbf{x}$  of  $\mathbb{R}^d/\Lambda$  with  $f$  a positive smooth function and  $\int_{\mathbb{R}^d/\Lambda} \mu = \deg_L(A)$ , there is a semi-positive metric  $\| \cdot \|$  on  $L$ , unique up to a constant multiple, such that  $c_1(\overline{L})^d = (i_A)_* \mu$ , where  $d\mathbf{x}$  is the Lebesgue measure on  $\mathbb{R}^d/\Lambda$  and  $\overline{L} = (L, \| \cdot \|)$ .*

The main ingredient of the proof is a limit formula for the measure associated to certain integrable metrics, which is Theorem 4.3. According to this formula, the existence part accounts to consider the following question in differential equation. We would like to mention it here since it is really interesting that we end up with a (real) Monge-Ampère equation on a real torus, quite similar to the complex case, although we are doing non-archimedean geometry. In fact, let  $\mathbb{R}^d/\Lambda$  be the real torus as above with usual coordinate  $x_1, \dots, x_d$  and Lebesgue measure  $d\mathbf{x} = dx_1 \cdots dx_d$ . Let  $(g_{ij})_{i,j=1,\dots,d}$  be a positive definite (real symmetric) matrix. Then for any smooth real function  $f$  on  $\mathbb{R}^d/\Lambda$  such that  $\int_{\mathbb{R}^d/\Lambda} e^f d\mathbf{x} = 1$ , we will show that there exists a unique smooth (real) function  $\phi$  such that:

- The matrix  $\left( g_{ij} + \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)$  is positive definite;
- $\int_{\mathbb{R}^d/\Lambda} \phi d\mathbf{x} = 0$ ;
- It satisfies the following real Monge-Ampère equation

$$\det \left( g_{ij} + \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \det(g_{ij}) \cdot e^f.$$

We will reduce it to the complex Monge-Ampère equation through an easy process. Then Yau's famous attack on this equation will imply our theorem. Hence the same PDE problem solves

this non-archimedean Calabi-Yau theorem as well!

At last, we would like make a remark about the notations. We use  $|\cdot|$  for the non-archimedean norm;  $\|\cdot\|$  for the metrics on line bundles. But due to the conventions, we will also use  $|\cdot|$  for the usual absolute value of real numbers and the total measure;  $\|\cdot\|$  for the Euclidean norm of  $\mathbb{R}^d$  when  $d > 1$ .

*Acknowledgements.* The paper is motivated by the work of Xinyi Yuan and Shou-Wu Zhang [16] on the uniqueness part of the Calabi-Yau theorem and Gubler [9] on the tropical geometry of totally degenerate abelian varieties. The author would also like to thank Xander Faber, Xinyi Yuan and Shou-Wu Zhang for useful discussion.

## 2 Mumford's construction

In this section, we briefly recall Mumford's construction of (formal) models of totally degenerate abelian varieties in [13, §6], also see [9, §4, §6].

*Valuation map.* Let  $k$  be the completion of the algebraic closure of a  $p$ -adic local field, for example,  $k = \mathbb{C}_p$ . Let  $|\cdot|$  be the norm on  $k$  and its extended valuation fields,  $k^\circ$  the sub-ring of  $k$  consisting of elements  $x \in k$  with  $|x| \leq 1$ ,  $k^{\circ\circ}$  the maximal ideal of  $k^\circ$  consisting of elements  $x \in k$  with  $|x| < 1$  and  $\tilde{k} = k^\circ/k^{\circ\circ}$  the residue field which is algebraically closed. We fix a logarithm  $\log$  such that  $\log|x| \in \mathbb{Q}$  for all  $x \in k$ .

Fix a split torus  $T = \mathbb{G}_{m,k}^d$  of rank  $d \geq 1$  over  $k$ . We have the following valuation map

$$\tau : T^{\text{an}} = \left( \mathbb{G}_{m,k}^d \right)^{\text{an}} \longrightarrow \mathbb{R}^d, \quad t \mapsto (-\log |T_1(t)|, \dots, -\log |T_d(t)|)$$

where  $\left( \mathbb{G}_{m,k}^d \right)^{\text{an}}$  is the associated  $k$ -analytic space (cf. [1, §3.4]) and  $T_i$  are coordinate functions. It is surjective and continuous with respect to the underlying topology of the analytic space and the usual topology of  $\mathbb{R}^d$ . And  $\tau$  has a continuous section

$$i : \mathbb{R}^d \longrightarrow T^{\text{an}}, \quad \mathbf{x} \mapsto \xi_{\mathbf{x}}$$

where  $\xi_{\mathbf{x}}$  is the Shilov boundary of the affinoid domain  $\tau^{-1}(\mathbf{x})$  (cf. [9, Corollary 4.5]).

Now consider a totally degenerate abelian variety  $A$  over  $k$ , i.e.,  $A^{\text{an}} \cong T^{\text{an}}/M$  for a complete lattice  $M \subset T(k)$ . By a complete lattice  $M$ , we mean that  $M$  bijectively maps to a rational complete lattice  $\Lambda \subset \mathbb{R}^d$  under  $\tau$ . Here, we say a complete lattice is rational if it has a basis whose coordinates are in  $\mathbb{Q}$ . Hence we have the induced map  $\tau_A : A^{\text{an}} \rightarrow \mathbb{R}^d/\Lambda$  and  $i_A : \mathbb{R}^d/\Lambda \hookrightarrow A^{\text{an}}$ . In fact,  $i_A$  identifies  $\mathbb{R}^d/\Lambda$  with a strong deformation retract or a skeleton of  $A^{\text{an}}$  as in [1, §6.5].

*Rational polytopes.* A compact subset  $\Delta$  of  $\mathbb{R}^d$  is called a *polytope* if it is an intersection of finitely many half-spaces  $\{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{m}_i \cdot \mathbf{x} \geq c_i\}$ . We say  $\Delta$  is *rational* if we can choose all  $\mathbf{m}_i \in \mathbb{Z}^d$  and  $c_i \in \mathbb{Q}$ . The dimension  $\dim(\Delta)$  of  $\Delta$  is its usual topological dimension and we denote by  $\text{int}(\Delta)$  the topological interior of  $\Delta$  in  $\mathbb{R}^d$ . A *closed face* of  $\Delta$  is either  $\Delta$  itself or  $B \cap \Delta$  where  $B$  is the boundary of a half-space containing  $\Delta$ . It is obvious that a closed face of a (rational) polytope is again a (rational) polytope. An *open face* is a closed face without its properly contained closed faces.

A (rational) polytopal complex  $\mathcal{C}$  in  $\mathbb{R}^d$  is a locally finite set of (rational) polytopes such that (1) if  $\Delta \in \mathcal{C}$ , then all its closed faces are in  $\mathcal{C}$  and (2) if  $\Delta, \Delta' \in \mathcal{C}$ , then  $\Delta \cap \Delta'$  is either empty or a closed face of both  $\Delta$  and  $\Delta'$ . The polytopes of dimension 0 are called *vertices*. We say  $\mathcal{C}$  is a (rational) polytopal decomposition of  $S \subset \mathbb{R}^d$  if  $S$  is the union of all polytopes in  $\mathcal{C}$ . In particular, if  $S = \mathbb{R}^d$ , we say  $\mathcal{C}$  is a (rational) polytopal decomposition of  $\mathbb{R}^d$ .

For a (rational) complete lattice  $\Lambda \subset \mathbb{R}^d$ , we say  $\mathcal{C}$  is  $\Lambda$ -periodic if  $\Delta \in \mathcal{C}$  implies  $\Delta + \lambda \in \mathcal{C}$  for all  $\lambda \in \Lambda$ . A (rational) polytopal decomposition  $\mathcal{C}_\Lambda$  of  $\mathbb{R}^d/\Lambda$  for a (rational) complete lattice  $\Lambda$  is a  $\Lambda$ -periodic (rational) polytopal decomposition  $\mathcal{C}$  of  $\mathbb{R}^d$  such that  $\Delta$  maps bijectively to its image under the projection  $\mathbb{R}^d \rightarrow \mathbb{R}^d/\Lambda$  for all  $\Delta \in \mathcal{C}$ . A polytope, a closed face or an open face of  $\mathcal{C}_\Lambda$  is a  $\Lambda$ -translation equivalence class of the corresponding object of  $\mathcal{C}$ .

A continuous real function  $f$  on  $\mathbb{R}^d$  is called (rational) polytopal if there is a (rational) polytopal decomposition  $\mathcal{C}$  of  $\mathbb{R}^d$  such that  $f$  restricted to all  $\Delta \in \mathcal{C}$  is affine (and takes rational values on all vertices). We denote by  $\mathcal{C}_{\text{poly}}(\mathbb{R}^d)$  ( $\mathcal{C}_{\text{rpoly}}(\mathbb{R}^d)$ ) the space of (rational) polytopal continuous functions on  $\mathbb{R}^d$ . We have the following simple lemma.

**Lemma 2.1.** *Let  $\Lambda$  be a (rational) complete lattice of  $\mathbb{R}^d$  and  $f$  in  $\mathcal{C}_{\text{poly}}(\mathbb{R}^d)$  ( $\mathcal{C}_{\text{rpoly}}(\mathbb{R}^d)$ ) satisfying*

(a) *There exist affine functions  $z_\lambda$  for all  $\lambda \in \Lambda$  satisfying  $f(\mathbf{x} + \lambda) = f(\mathbf{x}) + z_\lambda(\mathbf{x})$  for all  $\lambda$  and  $\mathbf{x} \in \mathbb{R}^d$ ;*

(b) *If  $\Delta$  is a maximal connected subset on which  $f$  is affine, then  $\Delta$  is a bounded convex subset of  $\mathbb{R}^d$ .*

*Then  $\Delta$  is a (rational) polytope and if  $\mathcal{C}$  is the polytopal complex generated by all such  $\Delta$  and their closed faces, then  $\mathcal{C}$  is a  $\Lambda$ -periodic (rational) polytopal decomposition of  $\mathbb{R}^d$ .*

*Proof.* Since  $\Delta$  is closed, hence compact by (b). Since  $f$  is (rational) polytopal, there is a finite (rational) polytopal decomposition of  $\Delta$ , hence  $\Delta$  itself is a (rational) polytope since it is convex. Let  $\mathcal{C}$  be the (rational) polytopal complex generated by all such  $\Delta$  and their closed faces which is obviously a decomposition of  $\mathbb{R}^d$ . The  $\Lambda$ -periodicity is implied by (a). □

*Formal models.* Let  $X$  be a projective scheme over  $k$ , a  $k^\circ$ -model  $\mathcal{X}$  of  $X$  is a scheme projective and flat over  $\text{Spec } k^\circ$  whose generic fibre  $\mathcal{X}_\eta \cong X$ . A formal  $k^\circ$ -model  $\mathcal{X}$  of  $X$  is an admissible formal scheme over  $\text{Spf } k^\circ$  whose generic fibre  $\mathcal{X}_\eta \cong X^{\text{an}}$ . We denote by  $\widetilde{\mathcal{X}}$  (resp.  $\widetilde{\mathcal{X}}$ ) the special fibre of  $\mathcal{X}$  (resp.  $\mathcal{X}$ ) which is a proper scheme over  $\text{Spec } \widetilde{k}$ . The following result is due to Mumford in the case  $A$  is the base change of an abelian variety over a  $p$ -adic local field (cf. [13, Corollary 6.6]) and generalized by Gubler in general case (cf. [9, Proposition 6.3]).

**Proposition 2.2.** *Given a rational polytopal decomposition  $\mathcal{C}_\Lambda$  of  $\mathbb{R}^d/\Lambda$ , we may associate a formal  $k^\circ$ -model  $\mathcal{A}$  of  $A$  whose special fibre  $\widetilde{\mathcal{A}}$  is reduced and the irreducible components  $Y$  of  $\widetilde{\mathcal{A}}$  are toric varieties and one-to-one correspond to the vertices  $\mathbf{v}$  of  $\mathcal{C}_\Lambda$  by  $\mathbf{v} = \tau_A(\xi_Y)$ , where  $\xi_Y \in A^{\text{an}}$  is the point corresponding to  $Y$ . The formal scheme  $\mathcal{A}$  has a covering by formal open affine sets  $\mathcal{U}_\Delta$  for  $\Delta \in \mathcal{C}_\Gamma$ . Moreover, if  $A$  is the base change of an abelian variety over a  $p$ -adic local field, then  $\mathcal{A}$  can be constructed as a  $k^\circ$ -model.*

### 3 Toric metrized line bundles

In this section, we briefly recall the theory of metrized line bundles and their associated measure for general varieties. We introduce line bundles and toric metrized line bundles on  $A$ . Then we

prove the main result identifying certain toric integrable metrics.

*Metrized line bundles and measure.* The general theory of metrized line bundles is developed in [17], [18], also see [5] and [8]. Let  $X$  be a projective scheme over  $k$  and  $L$  a line bundle over  $X$ . A *metric*  $\|\cdot\|$  on  $L$  is given that, for all open subset  $U$  of  $X^{\text{an}}$  and a section  $s \in \Gamma(U, L^{\text{an}})$ , a continuous function  $\|s\| : U \rightarrow \mathbb{R}_{\geq 0}$  such that  $\|fs\| = |f| \cdot \|s\|$  for all  $f \in \Gamma(U, \mathcal{O}_U)$  which is zero only if  $s = 0$ .

We say a metric is *algebraic* if it is defined by a model  $(\mathcal{X}, \mathcal{L})$  where  $\mathcal{X}$  is a  $k^\circ$ -model of  $X$  and  $\mathcal{L}$  is a line bundle on  $\mathcal{X}$  such that  $\mathcal{L}_\eta \cong L^e$  for some integer  $e \geq 1$ . A metric is *formal* if we replace the  $k^\circ$ -model  $\mathcal{X}$  by a formal  $k^\circ$ -model  $\mathcal{X}$  and  $\mathcal{L}$  by a formal line bundle  $\mathcal{L}$  on  $\mathcal{X}$  such that  $\mathcal{L}_\eta \cong (L^e)^{\text{an}}$ . In fact, all formal metrics are algebraic. An algebraic (resp. formal) metric is called *semi-positive* if the reduction  $\widetilde{\mathcal{L}}$  (resp.  $\widetilde{\mathcal{L}}$ ) has non-negative degree on all curves inside  $\widetilde{\mathcal{X}}$  (resp.  $\widetilde{\mathcal{X}}$ ). In general, a metric on  $L$  is called *semi-positive* if it is the uniform limit of algebraic semi-positive metrics. A metrized line bundle is called *integrable* if it is isomorphic to a quotient of two semi-positive metrized line bundles.

Next we recall the construction of measure by Chambert-Loir in [5, §2]. For simplicity, we only recall the algebraic case (which we only need for calculation later) and for general metric, one need to pass to the limit which we refer to *loc. cit.* for details. Let  $X$  be as above of dimension  $d \geq 1$  and  $L_i$  ( $i = 1, \dots, d$ ) line bundles on it. We endow  $L_i$  with an algebraic measure  $\|\cdot\|_i$  induced by  $(\mathcal{X}, \mathcal{L}_i)$  with  $(\mathcal{L}_i)_\eta \cong L_i^{e_i}$  on a common model  $\mathcal{X}$  which is assumed to be normal. Let  $Y_j$  be the reduced irreducible components of  $\widetilde{\mathcal{X}}$  and  $\xi_j$  the unique point in the inverse image of the generic point of  $Y_j$  under the reduction map  $\pi : X^{\text{an}} \rightarrow \widetilde{\mathcal{X}}$ . Then we define

$$c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d}) = \frac{1}{e_1 \cdots e_d} \sum_j m_j \left( c_1(\widetilde{\mathcal{L}_1}) \cdots c_1(\widetilde{\mathcal{L}_d})|_{Y_j} \right) \delta_{\xi_j} \quad (3.1)$$

where  $\overline{L_i} = (L_i, \|\cdot\|_i)$ ,  $m_j$  is the multiplicity of  $Y_j$  in  $\widetilde{\mathcal{X}}$  and  $\delta_{\xi_j}$  is the normalized Dirac measure supported at  $\xi_j$ . In general, the measure  $c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d})$  is symmetric and  $\mathbb{Z}$ -multi-linear and we have

$$\int_{X^{\text{an}}} c_1(\overline{L_1}) \wedge \cdots \wedge c_1(\overline{L_d}) = c_1(L_1) \cdots c_1(L_d)|_X. \quad (3.2)$$

*Line bundles on  $A$ .* The theory of line bundles on totally degenerate abelian varieties is very similar to that over complex field. We refer to [3, §2] and [6, Chapter 6] for more details.

Let  $A$  be a totally degenerate abelian variety as above and  $\check{M} = \text{Hom}_k(T, \mathbb{G}_{m,k})$  the character group of  $T$ . Let  $\check{T}$  be the split torus with character group  $M$ ; i.e.,  $\check{T} = \text{Hom}_k(M, \mathbb{G}_{m,k})$ . Then  $\check{A}^{\text{an}}$  is canonically isomorphic to  $\check{T}^{\text{an}}/\check{M}$  where  $\check{A}$  is the dual abelian variety of  $A$ . Let  $L$  be a line bundle on  $A$ , the pull-back of  $L$  to  $T$  is trivial and is identified with  $T \times \mathbb{G}_{a,k}$ . Hence  $L$  is identified with a quotient  $(T \times \mathbb{G}_{a,k})/M$  whose action is given by an element  $\mu \mapsto Z_\mu$  of  $H^1(M, \mathcal{O}(T)^\times)$ . The function  $Z_\mu$  has the form  $Z_\mu = d_\mu \sigma_\mu$  where  $d_\mu \in k^\times$ ,  $\mu \mapsto \sigma_\mu$  is a group homomorphism  $\sigma : M \rightarrow \check{M}$  and  $d_{\mu\nu} d_\mu^{-1} d_\nu^{-1} = \sigma_\nu(\mu)$ . By the isomorphism  $\tau : M \rightarrow \Lambda$ , we get a unique symmetric bilinear form  $b$  on  $\mathbb{R}^d$  such that  $b(\tau(\mu), \tau(\nu)) = -\log |\sigma_\nu(\mu)|$ . Then  $b$  is positive definite if and only if  $L$  is ample. And since  $\sigma_\mu$  is a character,  $Z_\mu$  factors through  $\tau$ , hence uniquely determines a function  $z_\lambda$  on  $\mathbb{R}^d$  such that  $z_\lambda(\tau(t)) = -\log |Z_\mu(t)|$  for all  $\mu \in M$  and  $t \in T$ , where  $\lambda = \tau(\mu)$ . The function  $z_\lambda$  is affine with

$$z_\lambda(\mathbf{x}) = z_\lambda(\mathbf{0}) + b(\mathbf{x}, \lambda), \quad \lambda \in \Lambda, \mathbf{x} \in \mathbb{R}^d. \quad (3.3)$$

Before stating the next lemma, we introduce some notations. We fix a  $\mathbb{Z}$ -basis  $(\lambda_1, \dots, \lambda_d)$  of  $\Lambda$  once for all and let  $\mathfrak{F} = \{\mathbf{x} = x_1\lambda_1 + \dots + x_d\lambda_d \mid 0 \leq x_i < 1\}$  be a fundamental domain of  $\Lambda$ . The volume of the closure  $\overline{\mathfrak{F}}$  under the usual Lebesgue measure  $d\mathbf{x}$  of  $\mathbb{R}^d$  only depends on  $\Lambda$  and will be denoted by  $\text{vol}(\Lambda)$ . We define  $R_{\mathfrak{F}} = \max_{\mathbf{x}, \mathbf{x}' \in \overline{\mathfrak{F}}} \|\mathbf{x} - \mathbf{x}'\|$  and  $r_{\mathfrak{F}}$  to be the maximal radius of balls contained in  $\overline{\mathfrak{F}}$ . We denote by  $S^{d-1} \subset \mathbb{R}^d$  the standard unit ball,  $\mathbf{SO}_d$  the special orthogonal group of  $\mathbb{R}^d$  and  $\mathfrak{S}_d$  the group of  $d$ -permutations.

Let  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$  be the standard base of the Euclidean space  $\mathbb{R}^d$ ,  $\mathcal{C}^l(\mathbb{R}^d)$  ( $l \geq 0$  and  $\mathcal{C} = \mathcal{C}^0$ ) the space of real functions whose  $l$ -th partial derivatives exist and are continuous,  $\mathcal{C}^\infty(\mathbb{R}^d)$  the space of real smooth functions and  $\mathcal{C}^{k,\alpha}(\mathbb{R}^d)$  ( $k \geq 0$ ,  $\alpha \in [0, 1)$ ) the spaces of real functions whose  $k$ -th partial derivatives exist and are Hölder continuous with exponential  $\alpha$ . We denote by  $\mathcal{C}_{\geq 0}^l(\mathbb{R}^d)$  the subspace of functions non-negative everywhere and  $\mathcal{C}_{> 0}^l(\mathbb{R}^d)$  that of functions positive everywhere; similarly, we have  $\mathcal{C}_{\geq 0}^\infty$ ,  $\mathcal{C}_{> 0}^\infty$ ,  $\mathcal{C}_{\geq 0}^{k,\alpha}$  and  $\mathcal{C}_{> 0}^{k,\alpha}$ . For a certain function  $f$ , we let  $f_i = \nabla_{\mathbf{e}_i} f$ ,  $f_{ij} = \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_j} f$  and  $f_{ijk} = \nabla_{\mathbf{e}_i} \nabla_{\mathbf{e}_j} \nabla_{\mathbf{e}_k} f$  if the corresponding directional derivatives exist.

Let  $q(\mathbf{x}) = \frac{1}{2}b(\mathbf{x}, \mathbf{x})$  be the associated quadratic form and  $H_q = d! \det(q_{ij})_{i,j=1,\dots,d}$  which is a constant. We have the following lemma.

**Lemma 3.1.** *If  $L$  is ample, then  $\deg_L(A) = \text{vol}(\Lambda)H_q$ .*

*Proof.* Consider the morphism  $\phi_L : A \rightarrow \tilde{A}$  associated to  $L$ . Its lifting  $T \rightarrow \tilde{T}$  restricts to  $\mu : M \rightarrow \tilde{M}$  on  $M$ . It is easy to see that  $\mu$  is injective since  $L$  is ample and

$$\deg(\mu) = [\tilde{M} : \mu(M)] = \text{vol}(\Lambda)^{-1} \det(b(\lambda_i, \lambda_j))_{i,j=1,\dots,d} = \text{vol}(\Lambda) \det(q_{ij})_{i,j=1,\dots,d}.$$

By [3, Theorem 6.15],  $\deg(\phi_L) = \deg(\mu)^2$  and by the Riemann-Roch theorem [12, §16],  $(\deg_L(A)/d!)^2 = \deg(\phi_L)$ . Since  $L$  is ample,  $\deg_L(A) > 0$  and hence equals  $\text{vol}(\Lambda)H_q$ . □

*Toric metrized line bundles.* The following proposition is due to Gubler.

**Proposition 3.2.** *Let  $L = (T \times \mathbb{G}_{a,k})/M$  be a line bundle on  $A$  given by a cocycle  $(Z_\mu)_{\mu \in M}$  as above. Let  $\mathcal{A}$  be a formal  $k^\circ$ -model determined by a rational polytopal decomposition  $\mathcal{C}_\Lambda$  of  $\mathbb{R}^d/\Lambda$  given by  $\mathcal{C}$ .*

(a) *There is a one-to-one correspondence between all formal metrics of  $L$ , i.e., formal model  $\mathcal{L}$  of  $L^e$  (with  $e$  minimal) on  $\mathcal{A}$ , with trivialization  $(\mathcal{U}_\Delta)_{\Delta \in \mathcal{C}_\Lambda}$  and functions  $g \in \mathcal{C}_{\text{rpoly}}(\mathbb{R}^d)$  satisfying*

$$g(\mathbf{x} + \lambda) = g(\mathbf{x}) + z_\lambda(\mathbf{x}); \quad \lambda \in \Lambda, \mathbf{x} \in \mathbb{R}^d. \quad (3.4)$$

*Moreover, if we denote by  $\|\cdot\|$  the corresponding formal metric on  $L$ , then we have*

$$g \circ \tau = -\log \circ p^* \|\cdot\| \quad (3.5)$$

*on  $T$ , where  $p : \mathbb{R}^d \rightarrow \mathbb{R}^d/\Lambda$  is the projection.*

(b) *The reduction  $\tilde{\mathcal{L}}$  is ample if and only if  $g$  is strongly polytopal convex with respect to  $\mathcal{C}$ , i.e., it is convex and the maximal connected subsets on which  $g$  is affine are  $\Delta \in \mathcal{C}$  with  $\dim(\Delta) = d$ .*

*Proof.* For (a), by (3.4) and the fact that there exist  $\mathbf{m}_\lambda \in \mathbb{Z}^d$  for all  $\lambda \in \Lambda$  such that  $b(\mathbf{x}, \lambda) = \mathbf{m}_\lambda \cdot \mathbf{x}$ , we can find a smallest integer  $e \geq 1$  such that  $e \cdot g$  has integer gradient everywhere. Then by [9, Proposition 6.6],  $e \cdot g$  determines a formal model  $\mathcal{L}$  of  $L^e$ . Hence  $g$  determines a formal metric on  $L$ . Then last identity (3.5) follows from *loc. cit.*

For (b), it follows from [9, Corollary 6.7]. □

**Definition 3.3.** Given  $L$  as above, we call a metric determined by the above proposition a *toric formal metric* and the corresponding function  $g$  the associated *formal Green function*. We denote by  $\mathcal{G}_{\text{for}}(L)$  the set of all formal Green functions of  $L$  and  $\mathcal{G}_+(L)$  the set of all uniform limits of formal Green functions of  $L$  associated to semi-positive toric formal metrics which we call *semi-positive Green functions*. It is easy to see that  $g \in \mathcal{G}_+(L)$  also satisfies (3.4) and the metric determined by (3.5) is semi-positive in its original sense. Similarly, we denote by  $\mathcal{G}_{\text{int}}(L)$  the set of deference of functions in  $\mathcal{G}_+(L')$  and  $\mathcal{G}_+(L'')$  with  $L = L' \otimes (L'')^{-1}$  which we call *integrable Green functions*, hence the corresponding metric is integrable. Moreover, the set of integrable Green functions for all line bundles  $L$  over  $A$ :  $\mathcal{G}_{\text{int}}(A) = \bigcup \mathcal{G}_{\text{int}}(L)$  is a torsion-free  $\mathbb{Z}$ -module. All metrized line bundles corresponding to the Green functions in  $\mathcal{G}_{\text{int}}(A)$  all called *toric metrized line bundles*. At last we denote by  $\mathcal{G}(L)$  the set of continuous real functions satisfying (3.4) and we simply call them *Green functions* of  $L$ .

The following proposition provides a certain large class of semi-positive Green functions for an ample line bundle.

**Proposition 3.4.** *Let  $L$  be an ample line bundle and  $g \in \mathcal{G}(L) \cap \mathcal{C}^2(\mathbb{R}^d)$  such that the matrix  $(g_{ij}(\mathbf{x}))_{i,j=1,\dots,d}$  is semi-positive definite everywhere, then  $g$  is inside  $\mathcal{G}_+(L)$ .*

*Proof.* The proof is divided into several steps.

*Step 1.* We reduce to the case where  $(g_{ij}(\mathbf{x}))_{i,j=1,\dots,d}$  is positive definite for all  $\mathbf{x} \in \mathbb{R}^d$ .

First, there exists a function  $g_{\text{can}} \in \mathcal{G}(L) \cap \mathcal{C}^2(\mathbb{R}^d)$  such that the matrix  $((g_{\text{can}})_{ij}(\mathbf{x}))$  is positive definite. By [6, Lemma 6.5.2 (4)], there is a group homomorphism  $c : \Lambda \rightarrow \mathbb{Q}$  such that  $z_{\Lambda}(\mathbf{0}) = q(\lambda) + c(\lambda)$ . We linearly extend  $c$  to  $\mathbb{R}^d$  and define  $g_{\text{can}} = q + c$ , then  $g_{\text{can}} \in \mathcal{G}(L) \cap \mathcal{C}^2(\mathbb{R}^d)$  and  $((g_{\text{can}})_{ij}(\mathbf{x}))$  is a constant positive definite matrix. Next, for any  $g \in \mathcal{G}(L) \cap \mathcal{C}^2(\mathbb{R}^d)$ ,  $g_{ij}$  is  $\Lambda$ -periodic for any  $(i, j)$  and  $g - g_{\text{can}}$  is a  $\Lambda$ -periodic  $\mathcal{C}^2$ -function. For any  $g$  in the proposition, let  $f = g - g_{\text{can}}$  and  $g_t = g_{\text{can}} + tf$  for  $t \in [0, 1]$ , then  $g_t \rightarrow g_1 = g$  when  $t \rightarrow 1$  and  $((g_t)_{ij}(\mathbf{x}))$  are positive definite for all  $t < 1$ . The claim follows.

*Step 2.* Now fix a function  $g$  as in the proposition but with the condition that  $(g_{ij}(\mathbf{x}))_{i,j=1,\dots,d}$  is positive definite, we are going to construct a sequence of functions  $g_n \in \mathcal{G}_{\text{for}}(L) \cap \mathcal{G}_+(L)$  approaching  $g$ . For any  $\mathbf{u} \in S^{d-1}$ , the function  $\nabla_{\mathbf{u}} \nabla_{\mathbf{u}} g$  is  $\Lambda$ -periodic and positive, hence there exist  $0 < h_g < H_g$  such that  $h_g < \nabla_{\mathbf{u}} \nabla_{\mathbf{u}} g(\mathbf{x}) < H_g$  for any  $\mathbf{u} \in S^{d-1}$  and  $\mathbf{x} \in \mathbb{R}^d$ .

Let  $N$  be a sufficiently large integer, for  $\mathbf{j} = (j_1, \dots, j_d) \in (\frac{1}{N}\mathbb{Z})^d$ , we let  $\lambda_{\mathbf{j}} = j_1 \lambda_1 + \dots + j_d \lambda_d$ . For each  $\mathbf{j}$  such that  $\lambda_{\mathbf{j}} \in \mathfrak{F}$ , we choose a positive number  $\epsilon_N(\mathbf{j}) < \frac{1}{N^2}$  such that  $g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j}) \in \mathbb{Q}$  and a vector  $\epsilon_N(\mathbf{j})$  such that  $\|\epsilon_N(\mathbf{j})\| < \frac{1}{N}$ ,  $\nabla g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j}) \in \mathbb{Q}^d$  and the graph of the function

$$g_{\lambda_{\mathbf{j}}}^{(N)}(\mathbf{x}) = (\nabla g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j}))(\mathbf{x} - \lambda_{\mathbf{j}}) + g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j})$$

is below the graph of  $g$  which is possible since  $g$  is strictly convex. For general  $\mathbf{j}$ , we let  $\mathbf{j}_0$  be the unique element in  $(\frac{1}{N}\mathbb{Z})^d$  such that  $\mathbf{j} - \mathbf{j}_0 \in \mathbb{Z}^d$  and  $\lambda_{\mathbf{j}_0} \in \mathfrak{F}$ . Then we define

$$g_{\lambda_{\mathbf{j}}}^{(N)}(\mathbf{x}) = (\nabla g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j}_0))(\mathbf{x} - \lambda_{\mathbf{j}}) + g(\lambda_{\mathbf{j}}) - \epsilon_N(\mathbf{j}_0).$$

By construction,  $g_{\lambda_{\mathbf{j}}}^{(N)} \in \mathcal{C}_{\text{rpol}}(\mathbb{R})$ . We need the following lemma.

**Lemma 3.5.** For each  $\mathbf{j} \in (\frac{1}{N}\mathbb{Z})^d$  and  $\boldsymbol{\lambda} \in \Lambda$ , we have

$$g(\mathbf{x}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x}) = g(\mathbf{x} + \boldsymbol{\lambda}) - g_{\boldsymbol{\lambda}_{\mathbf{j}} + \boldsymbol{\lambda}}^{(N)}(\mathbf{x} + \boldsymbol{\lambda}).$$

*Proof.* By definition, we have

$$\begin{aligned} & g_{\boldsymbol{\lambda}_{\mathbf{j}} + \boldsymbol{\lambda}}^{(N)}(\mathbf{x} + \boldsymbol{\lambda}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x}) \\ &= (\nabla g(\boldsymbol{\lambda}_{\mathbf{j}} + \boldsymbol{\lambda}) - \nabla g(\boldsymbol{\lambda}_{\mathbf{j}}))(\mathbf{x} - \boldsymbol{\lambda}_{\mathbf{j}}) + g(\boldsymbol{\lambda}_{\mathbf{j}} + \boldsymbol{\lambda}) - g(\boldsymbol{\lambda}_{\mathbf{j}}) \\ &= \nabla z_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{\mathbf{j}})(\mathbf{x} - \boldsymbol{\lambda}_{\mathbf{j}}) + z_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{\mathbf{j}}) \\ &= z_{\boldsymbol{\lambda}}(\mathbf{x}) \\ &= g(\mathbf{x} + \boldsymbol{\lambda}) - g(\mathbf{x}) \end{aligned}$$

where the third equality is because  $z_{\boldsymbol{\lambda}}$  is affine. □

*Step 3.* We define a function  $g^{(N)}$  by

$$g^{(N)}(\mathbf{x}) = \sup_{\mathbf{j}} g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x})$$

which is less than  $g(\mathbf{x})$ . We have

**Lemma 3.6.** For any compact subset  $V \subset \mathbb{R}^d$ , there exists a finite subset  $\mathcal{J}_V \subset (\frac{1}{N}\mathbb{Z})^d$  such that

$$g^{(N)}(\mathbf{x}) = \max_{\mathbf{j} \in \mathcal{J}_V} g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x})$$

for all  $\mathbf{x} \in V$ .

*Proof.* We only need to prove that for given  $M \in \mathbb{R}$ , there are only finitely many  $\mathbf{j}$  such that  $g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x}) \geq M$  for some  $\mathbf{x} \in V$ . For a given  $\mathbf{j}$ , we try to give a lower bound for the difference  $g(\mathbf{x}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x})$ . Let  $\mathbf{u} = \mathbf{x} - \boldsymbol{\lambda}_{\mathbf{j}}$ , by definition,

$$\begin{aligned} & g(\mathbf{x}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x}) \\ &> \left( g(\mathbf{x}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\mathbf{x}) \right) - \left( g(\boldsymbol{\lambda}_{\mathbf{j}}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\boldsymbol{\lambda}_{\mathbf{j}}) \right) \\ &= \int_0^1 \frac{d}{dt} \left( g(\boldsymbol{\lambda}_{\mathbf{j}} + t\mathbf{u}) - g_{\boldsymbol{\lambda}_{\mathbf{j}}}^{(N)}(\boldsymbol{\lambda}_{\mathbf{j}} + t\mathbf{u}) \right) dt \\ &= \int_0^1 \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}} + t\mathbf{u}) dt - \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}}) - \boldsymbol{\epsilon}_N(\mathbf{j}_0) \cdot \mathbf{u} \\ &= \int_0^1 \left( \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}}) + \int_0^t \nabla_{\mathbf{u}} \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}} + s\mathbf{u}) ds \right) dt - \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}}) - \boldsymbol{\epsilon}_N(\mathbf{j}_0) \cdot \mathbf{u} \\ &= \int_0^1 \int_0^t \nabla_{\mathbf{u}} \nabla_{\mathbf{u}} g(\boldsymbol{\lambda}_{\mathbf{j}} + s\mathbf{u}) ds dt - \boldsymbol{\epsilon}_N(\mathbf{j}_0) \cdot \mathbf{u} \\ &> \frac{h_g \|\mathbf{u}\|^2}{2} - \|\boldsymbol{\epsilon}_N(\mathbf{j}_0)\| \cdot \|\mathbf{u}\| \\ &> \frac{h_g \|\mathbf{u}\|^2}{2} - \frac{\|\mathbf{u}\|}{N}. \end{aligned}$$



We see that there is  $N_M > 0$  such that  $\|\lambda_j - \mathbf{x}\| > N_M$  implies  $g_{\lambda_j}^{(N)}(\mathbf{x}) < M$ . Hence the lemma follows.  $\square$

The above lemma implies that  $g^{(N)} \in \mathcal{C}_{\text{poly}}(\mathbb{R}^d)$  and is convex. On the other hand, if  $g^{(N)}(\mathbf{x}) = g_{\lambda_j}^{(N)}(\mathbf{x})$  for some  $\mathbf{j}$ , then  $g^{(N)}(\mathbf{x} + \lambda) = g_{\lambda_j + \lambda}^{(N)}(\mathbf{x} + \lambda)$  for all  $\lambda \in \Lambda$  since by Lemma 3.5,  $g_{\lambda_{j'}}^{(N)}(\mathbf{x} + \lambda) > g_{\lambda_j + \lambda}^{(N)}(\mathbf{x} + \lambda)$  will imply that  $g_{\lambda_{j'} - \lambda}^{(N)}(\mathbf{x}) > g_{\lambda_j}^{(N)}(\mathbf{x})$  which is a contradiction. Again by the same lemma, we conclude that  $g^{(N)}$  satisfies (3.4), hence is inside  $\mathcal{G}(L)$ .

*Step 4.* Before proving that  $g^{(N)}$  is semi-positive formal, we would like bound the difference of it and  $g$ .

**Lemma 3.7.** *For any  $\mathbf{x} \in \mathbb{R}^d$ , we have*

$$0 \leq g(\mathbf{x}) - g^{(N)}(\mathbf{x}) < \frac{R_{\mathfrak{F}}^2 \cdot H_g + 2R_{\mathfrak{F}} + 2}{2N^2}.$$

*Proof.* The proof follows the same line as in Lemma 3.6. Hence we have

$$g(\mathbf{x}) - g^{(N)}(\mathbf{x}) < \frac{H_g}{2} \|\lambda_j - \mathbf{x}\|^2 + \frac{1}{N} \|\lambda_j - \mathbf{x}\| + \frac{1}{N^2}$$

for any  $\mathbf{j}$ . In fact, we can choose  $\mathbf{j}$  such that  $\|\lambda_j - \mathbf{x}\| \leq \frac{R_{\mathfrak{F}}}{N}$ . Hence the lemma follows.  $\square$

Conversely, we have the following lemma on the estimate of the gradient which will be used also in the next section.

**Lemma 3.8.** *Let  $f$  be any convex rational polytopal continuous function on  $\mathbb{R}^d$ . Suppose that  $|f(\mathbf{x}) - g(\mathbf{x})| < \epsilon$  for any  $\mathbf{x} \in \mathbb{R}^d$ , then for any  $d$ -dimensional polytope  $\Delta$  on which  $f$  is affine,  $\|\mathbf{m}_{\Delta} - \nabla g(\mathbf{x}_0)\| \leq 2\sqrt{\epsilon H_g}$  for all  $\mathbf{x}_0 \in \Delta$ , where  $\mathbf{m}_{\Delta}$  is the gradient of  $f$  on  $\Delta$ .*

*Proof.* By continuity, we can assume that  $\mathbf{x}_0 \in \text{int}(\Delta)$  and  $\mathbf{x}_0 = \mathbf{0}$ . We only need to prove the lemma for  $\tilde{f}$  and  $\tilde{g}$  where  $\tilde{f}(\mathbf{x}) = f(\mathbf{x}) - \nabla g(\mathbf{0}) \cdot \mathbf{x} - g(\mathbf{0})$  and  $\tilde{g}(\mathbf{x}) = g(\mathbf{x}) - \nabla g(\mathbf{0}) \cdot \mathbf{x} - g(\mathbf{0})$ . In this case, we need to prove that  $\|\mathbf{m}_{\Delta}\| < 2\sqrt{\epsilon H_g}$ . For any  $\mathbf{u} \in S^{d-1}$ , we assume that  $\mathbf{m}_{\Delta} \cdot \mathbf{u} \geq 0$ ; otherwise, we take  $-\mathbf{u}$ . For  $t > 0$ , consider

$$\begin{aligned} & \left( \tilde{f}(t\mathbf{u}) - \tilde{g}(t\mathbf{u}) \right) - \left( \tilde{f}(\mathbf{0}) - \tilde{g}(\mathbf{0}) \right) \\ &= \int_0^t \left( \nabla_{\mathbf{u}} \tilde{f}(s\mathbf{u}) - \nabla_{\mathbf{u}} \tilde{g}(s\mathbf{u}) \right) ds \\ &\geq \mathbf{m}_{\Delta} \cdot \mathbf{u} t - \int_0^t \int_0^s \nabla_{\mathbf{u}} \nabla_{\mathbf{u}} \tilde{g}(r\mathbf{u}) dr ds \\ &> \mathbf{m}_{\Delta} \cdot \mathbf{u} t - \frac{H_g t^2}{2} \end{aligned}$$

where the first inequality is due to the assumption that  $f$  is convex. On the other hand, it is less than  $2\epsilon$ , hence we have

$$\frac{H_g t^2}{2} - \mathbf{m}_{\Delta} \cdot \mathbf{u} t + 2\epsilon > 0$$

for all  $t > 0$ . Hence  $\mathbf{m}_{\Delta} \cdot \mathbf{u} < 2\sqrt{\epsilon H_g}$  and then  $\|\mathbf{m}_{\Delta}\| \leq 2\sqrt{\epsilon H_g}$ .  $\square$

The above lemma immediately implies the following

**Lemma 3.9.** *Let  $f$  and  $\Delta$  be as above, then for any  $\mathbf{x}, \mathbf{x}' \in \Delta$ , the distance  $\|\mathbf{x} - \mathbf{x}'\| \leq \frac{4}{h_g} \sqrt{\epsilon H_g}$ .*

In particular, if we apply the above lemma to  $g^{(N)}$ , we see that  $\Delta$  is compact for any maximal connected subset  $\Delta$  on which  $g^{(N)}$  is affine.

*Step 5.* We would like to apply Lemma 2.1. Hence we need to show that  $\Delta$  is convex. By construction,  $g^{(N)}$  coincides with some  $g_{\lambda_j}^{(N)}$  restricted to  $\Delta$ . Suppose that there are  $\mathbf{x}_0, \mathbf{x}_1 \in \Delta$  and  $t \in (0, 1)$  such that  $\mathbf{x}_t = t\mathbf{x}_1 + (1-t)\mathbf{x}_0 \notin \Delta$ , then  $g^{(N)}(\mathbf{x}_t) = g_{\lambda_{j'}}^{(N)}(\mathbf{x}_t)$  for some  $j' \neq j$ . Again by construction,  $g_{\lambda_{j'}}^{(N)}(\mathbf{x}_t) > g_{\lambda_j}^{(N)}(\mathbf{x}_t)$ . Hence there is one point  $\mathbf{x} \in \{\mathbf{x}_0, \mathbf{x}_1\}$  such that  $g_{\lambda_{j'}}^{(N)}(\mathbf{x}) > g_{\lambda_j}^{(N)}(\mathbf{x})$  which is a contradiction. Now by Lemma 2.1,  $g^{(N)}$  determines a  $\Lambda$ -periodic rational polytopal decomposition  $\mathcal{C}$  of  $\mathbb{R}^d$ .

Finally, we prove that for any  $\Delta \in \mathcal{C}$ ,  $\Delta$  maps bijectively to its image under the projection  $p : \mathbb{R}^d \rightarrow \mathbb{R}^d/\Lambda$  when  $N$  is sufficiently large. We can assume  $\dim(\Delta) = d$ . By Lemma 3.7 and 3.9, we see that this holds if

$$\frac{4}{h_g} \sqrt{\frac{R_{\mathfrak{F}}^2 \cdot H_g + 2R_{\mathfrak{F}} + 2}{2N^2} \cdot H_g} < 2r_g.$$

Now by Proposition 3.2,  $g^{(N)}$  is a semi-positive formal Green function for large  $N$ . Hence the proposition follows by Lemma 3.7.  $\square$

The proposition has the following direct corollary.

**Corollary 3.10.** *For any line bundle  $L$  on  $A$ , if  $g \in \mathcal{G}(L) \cap \mathcal{C}^2(\mathbb{R}^d)$ , then  $g$  is integrable, i.e.,  $g \in \mathcal{G}_{\text{int}}(L)$ .*

## 4 A limit formula for the measure

In this section, we prove a formula for the measure of metrics determined by certain integrable Green functions.

*Measures on torus and mixed Hessian.* Recall that we have a closed manifold  $\mathbb{R}^d/\Lambda$ . Similar to  $\mathbb{R}^d$ , we define the space of real functions  $\mathcal{C}^l(\mathbb{R}^d/\Lambda)$ ,  $\mathcal{C}^\infty(\mathbb{R}^d/\Lambda)$ ,  $\mathcal{C}^{k,\alpha}(\mathbb{R}^d/\Lambda)$  and also  $\mathcal{C}_{\geq 0}^l(\mathbb{R}^d/\Lambda)$ ,  $\mathcal{C}_{> 0}^l(\mathbb{R}^d/\Lambda)$ . A *measure* on  $\mathbb{R}^d/\Lambda$  is a continuous linear functional  $\mu : \mathcal{C}(\mathbb{R}^d/\Lambda) \rightarrow \mathbb{R}$ ; it is *semi-positive* if  $\mu(f) \geq 0$  for all  $f \in \mathcal{C}_{\geq 0}(\mathbb{R}^d/\Lambda)$ ; *positive* if  $\mu(f) > 0$  for  $0 \neq f \in \mathcal{C}_{\geq 0}(\mathbb{R}^d/\Lambda)$ . The space of all measure (resp. semi-positive measure, positive measure) is denoted by  $\mathcal{M}(\mathbb{R}^d/\Lambda)$  (resp.  $\mathcal{M}_{\geq 0}(\mathbb{R}^d/\Lambda)$ ,  $\mathcal{M}_{> 0}(\mathbb{R}^d/\Lambda)$ ). It is endowed with the weak topology; i.e., a sequence  $\mu_n \rightarrow \mu$  if and only if  $\mu_n(f) \rightarrow \mu(f)$  for all  $f \in \mathcal{C}(\mathbb{R}^d/\Lambda)$ . Recall that we have the Lebesgue measure  $d\mathbf{x}$  on  $\mathbb{R}^d/\Lambda$ , hence the spaces of functions  $\mathcal{C}^l(\mathbb{R}^d/\Lambda)$  can be identified as a subset of measure by integration which we denote by  $\mathcal{M}^l(\mathbb{R}^d/\Lambda)$  for  $l = l; \infty; k, \alpha$ . Under this identification, being semi-positive (non-negative) or positive for a function coincides with that for a measure. Hence we also introduce the notation  $\mathcal{M}_{\geq 0}^l(\mathbb{R}^d/\Lambda)$  or  $\mathcal{M}_{> 0}^l(\mathbb{R}^d/\Lambda)$  for their obvious meaning. We write  $\mu \leq \mu'$  if  $\mu' - \mu \in \mathcal{M}_{\geq 0}(\mathbb{R}^d/\Lambda)$ . Finally, we denote by  $|\mu|$  the totally measure of  $\mu$ , i.e.,  $|\mu| = \mu(1)$ . It is easy to see that if  $\mu$  is semi-positive and  $|\mu| = 0$ , then  $\mu = 0$ .

**Definition 4.1.** For functions  $g_1, \dots, g_d \in \mathcal{C}^2(\mathbb{R}^d)$ , we define the so-called *mixed Hessian* of  $g_1, \dots, g_d$  to be the function

$$\text{Hess}_{g_1, \dots, g_d}(\mathbf{x}) = \sum_{\mu, \nu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu\nu)} \prod_{i=1}^d (g_i)_{\mu(i)\nu(i)}(\mathbf{x})$$

where  $\epsilon(\mu) = 1$  (resp.  $-1$ ) if  $\mu$  is an even (resp. odd) permutation. In particular, when  $g_1 = \dots = g_d = g$ ,

$$\text{Hess}_g(\mathbf{x}) := \text{Hess}_{g, \dots, g}(\mathbf{x}) = d! \det(g_{ij})_{i,j=1, \dots, d}$$

is  $d!$  times the determinant of the usual Hessian matrix of  $g$ .

The following lemma is an easy calculus exercise.

**Lemma 4.2.** Let  $L$  be an ample line bundle on  $A$  and let  $g_1, \dots, g_d \in \mathcal{G}(L) \cap \mathcal{C}^3(\mathbb{R}^d)$ , then  $\text{Hess}_{g_1, \dots, g_d}$  is  $\Lambda$ -periodic and hence in  $\mathcal{C}^1(\mathbb{R}^d/\Lambda)$ . We have

$$\int_{\mathbb{R}^d/\Lambda} \text{Hess}_{g_1, \dots, g_d}(\mathbf{x}) d\mathbf{x} = \deg_L(A).$$

*Proof.* First, we check the case  $g_1 = \dots = g_d = g_{\text{can}}$ . Then

$$\int_{\mathbb{R}^d/\Lambda} \text{Hess}_{g_{\text{can}}}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d/\Lambda} H_q d\mathbf{x} = \text{vol}(\Lambda) H_q$$

which equals  $\deg_L(A)$  by Lemma 3.1. In general, since any two  $g, g' \in \mathcal{G}(L) \cap \mathcal{C}^3(\mathbb{R}^d)$  differ by a  $\Lambda$ -periodic function  $f \in \mathcal{C}^3(\mathbb{R}^d)$ , by multi-linearity and symmetry, we only need to prove that for such  $f$  and  $g_2, \dots, g_d \in \mathcal{G}(L) \cap \mathcal{C}^3(\mathbb{R}^d)$ ,

$$\int_{\mathbb{R}^d/\Lambda} \text{Hess}_{f, g_2, \dots, g_d}(\mathbf{x}) d\mathbf{x} = 0. \quad (4.1)$$

By a linear change of coordinates, one can assume that  $\lambda_i = \mathbf{e}_i$  for  $i = 1, \dots, d$ . Hence (4.1) becomes

$$\int_{[0,1]^d} \text{Hess}_{f, g_2, \dots, g_d}(\mathbf{x}) dx_1 \cdots dx_d = 0.$$

For each  $i = 1, \dots, d$ , let us denote by

$$F_i^+ = \{\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d \mid x_i = 1\}; \quad F_i^- = \{\mathbf{x} = (x_1, \dots, x_d) \in [0, 1]^d \mid x_i = 0\}.$$

Then

$$\begin{aligned} & \int_{[0,1]^d} \text{Hess}_{f, g_2, \dots, g_d}(\mathbf{x}) dx_1 \cdots dx_d \\ &= \int_{[0,1]^d} \sum_{\mu, \nu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu\nu)} f_{\mu(1)\nu(1)} \prod_{i=2}^d (g_i)_{\mu(i)\nu(i)} dx_1 \cdots dx_d \\ &= \sum_{\mu, \nu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu\nu)} \left( \int_{F_{\mu(1)}^+} f_{\nu(1)} \prod_{i=2}^d (g_i)_{\mu(i)\nu(i)} d\mathbf{y} - \int_{F_{\mu(1)}^-} f_{\nu(1)} \prod_{i=2}^d (g_i)_{\mu(i)\nu(i)} d\mathbf{y} - S_{\mu, \nu} \right) \\ &= - \sum_{\mu, \nu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu\nu)} S_{\mu, \nu} \end{aligned}$$

since  $f$  and  $(g_i)_{\mu(i)\nu(i)}$  are  $\Lambda$ -periodic, where

$$S_{\mu,\nu} = \int_{[0,1]^d} f_{\nu(1)} \sum_{j=2}^d (g_j)_{\mu(1)\mu(j)\nu(j)} \prod_{\substack{i=2 \\ i \neq j}}^d (g_i)_{\mu(i)\nu(i)} dx_1 \cdots dx_d.$$

But then

$$\begin{aligned} & \sum_{\mu,\nu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu\nu)} S_{\mu,\nu} \\ &= \sum_{j=2}^d \sum_{\nu \in \mathfrak{S}_d} (-1)^{\epsilon(\nu)} \int_{[0,1]^d} f_{\nu(1)} \left( \sum_{\mu \in \mathfrak{S}_d} (-1)^{\epsilon(\mu)} (g_j)_{\mu(1)\mu(j)\nu(j)} \prod_{\substack{i=2 \\ i \neq j}}^d (g_i)_{\mu(i)\nu(i)} \right) dx_1 \cdots dx_d \\ &= 0 \end{aligned}$$

since in the inner summation, the terms with  $\mu$  and  $\mu'$  such that  $\mu'(1) = \mu(j)$ ,  $\mu'(j) = \mu(1)$  and  $\mu'(i) = \mu(i)$  for  $i \neq 1, j$  cancel each other. Hence the lemma follows.  $\square$

*Limit formula for the measure.* Before we state the formula, we would like to introduce the notion of dual polytopes. The main reference for this is [11, §2, §3]. Let  $\mathcal{C}$  be a (rational) polytopal decomposition of  $\mathbb{R}^d$  and  $f$  be a (rational) polytopal function, strongly polytopal convex with respect to  $\mathcal{C}$ . Then for any vertex  $\mathbf{v} \in \mathcal{C}$ , let  $\text{star}(\mathbf{v})$  be the set of all  $d$ -dimensional polytopes  $\Delta \in \mathcal{C}$  containing  $\mathbf{v}$  and for such  $\Delta$ , let  $\mathbf{m}_\Delta$  be the gradient of  $f$  on  $\Delta$  (which is just the *peg* in *loc. cit.*). Then the *dual polytopal* of  $\mathbf{v}$  with respect to  $f$  is defined to be the convex hull of points  $\mathbf{m}_\Delta$  for all  $\Delta \in \text{star}(\mathbf{v})$ , which we denote by  $\widehat{\mathbf{v}}^f$ . It is a  $d$ -dimensional (rational) polytope.

Now we are going to prove the following main theorem of this section. Recall that we have the embedding  $i_A : \mathbb{R}^d/\Lambda \hookrightarrow A^{\text{an}}$ .

**Theorem 4.3.** *Let  $\overline{L}_i = (L_i, \|\cdot\|_i)$  ( $i = 1, \dots, d$ ) be  $d$  integrable metrized line bundles on  $A$ , where  $\|\cdot\|_i$  are toric determined by Green functions  $g_i \in \mathcal{G}_{\text{int}}(L_i) \cap \mathcal{C}^3(\mathbb{R}^d)$ . Then we have the following equality of measures on  $A^{\text{an}}$ :*

$$c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_d) = (i_A)_* \text{Hess}_{g_1, \dots, g_d} d\mathbf{x}.$$

*Proof.* The proof is divided into several steps.

*Step 1.* First we reduce to the case  $\overline{L}_1 = \cdots = \overline{L}_d = \overline{L} = (L, \|\cdot\|)$  where  $L$  is an ample line bundle on  $A$  and  $\|\cdot\|$  is determined by a Green function  $g \in \mathcal{G}_+(L) \cap \mathcal{C}^3(\mathbb{R}^d)$  such that the matrix  $(g_{ij})_{i,j=1,\dots,d}$  is positive-definite everywhere. Assuming this, consider the subset  $\mathcal{G}'(A) \subset \mathcal{G}_{\text{int}}(A)$  consisting of such Green functions. Then  $g, g' \in \mathcal{G}'(A)$  implies  $ag + bg' \in \mathcal{G}'(A)$  for  $(a, b) \in \mathbb{Z}_{\geq 0}^2 - \{(0, 0)\}$ . For any continuous function  $f$  on  $A^{\text{an}}$ , consider the functional

$$\ell_f(g_1, \dots, g_d) = (c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_d) - (i_A)_* \text{Hess}_{g_1, \dots, g_d} d\mathbf{x})(f)$$

which is symmetric and  $\mathbb{Z}$ -multi-linear in  $g_1, \dots, g_d$  and  $\ell_f(g, \dots, g) = 0$  for  $g \in \mathcal{G}'(A)$  by our assumption. Then for  $g_1, \dots, g_d \in \mathcal{G}'(A)$  and  $t_1, \dots, t_d \in \mathbb{Z}_{>0}^d$ ,

$$\begin{aligned} 0 &= \ell_f \left( \sum_{i=1}^d t_i g_i, \dots, \sum_{i=1}^d t_i g_i \right) \\ &= \sum_{\substack{k_1, \dots, k_d \geq 0 \\ k_1 + \dots + k_d = d}} \frac{d!}{k_1! \dots k_d!} \ell_f(\dots, g_i, \dots, g_i, \dots) t_1^{k_1} \dots t_d^{k_d} \end{aligned}$$

where  $g_i$  appears  $d_i$  times in the second  $\ell_f$ . Hence  $\ell_f(\dots, g_i, \dots, g_i, \dots) = 0$ . In particular,  $\ell_f(g_1, \dots, g_d) = 0$  for all  $g_i \in \mathcal{G}'(A)$ . But on the other hand,  $\mathcal{G}'(A)$  generates the whole space  $\mathcal{G}_{\text{int}}(A) \cap \mathcal{C}^3(\mathbb{R}^d)$  by definition and the existence and smoothness of  $g_{\text{can}}$ . Then  $\ell_f(g_1, \dots, g_d) = 0$  for all  $g_i \in \mathcal{G}_{\text{int}}(A) \cap \mathcal{C}^3(\mathbb{R}^d)$ . The theorem follows.

*Step 2.* Now fix  $g \in \mathcal{G}'(A)$  as above and assume  $g \in \mathcal{G}_+(L)$  for an ample line bundle  $L$ . By Proposition 3.2, there are  $g_n \in \mathcal{G}_{\text{for}}(L)$  such that  $g_n \rightarrow g$  and the corresponding formal  $k^\circ$ -models  $(\mathcal{X}_n, \mathcal{L}_n)$  satisfying that  $\widetilde{\mathcal{L}}_n$  is ample on  $\widetilde{\mathcal{X}}_n$ . Hence the models are in fact algebrizable. Now we view  $\mathcal{X}_n$  as schemes projective and flat over  $\text{Spec } k^\circ$  and  $(\mathcal{L}_n)_\eta \cong L^\eta$ . We denote the corresponding metrized line bundle determined by  $g_n$  by  $\overline{L}_n = (L, \|\cdot\|_n)$  and by  $g$  by  $\overline{L} = (L, \|\cdot\|)$ . By (3.1) and Proposition 2.2, the measure  $c_1(\overline{L}_n)^{\wedge d}$  is supported on  $i_A(\mathbb{R}^d/\Lambda)$  hence also for their limit. If we let  $\mu_n$  be the measure  $c_1(\overline{L}_n)^{\wedge d}$  restricted on  $\mathbb{R}^d/\Lambda$  and  $\mu = \lim_n \mu_n$ . Then we only need to prove that  $\mu = \mu_g$  as elements in  $\mathcal{M}(\mathbb{R}^d/\Lambda)$ , where  $\mu_g$  is the measure  $\text{Hess}_g dx \in \mathcal{M}_{>0}^3(\mathbb{R}^d/\Lambda)$ .

We claim that for any  $\delta > 0$ , we have

$$\mu \leq (1 + \delta) \mu_g. \quad (4.2)$$

Assuming this, then  $\mu \leq \mu_g$ . But by (3.2) and Lemma 4.2,  $|\mu| = |\mu_g|$ ; i.e.,  $|\mu_g - \mu| = 0$ . Hence  $\mu = \mu_g$  which confirms the theorem.

*Step 3.* The last step is dedicated to prove the above claim (4.2). We prove that for any  $\delta > 0$  and  $f \in \mathcal{C}_{\geq 0}(\mathbb{R}^d/\Lambda)$ ,  $\mu(f) \leq (1 + \delta) \mu_g(f)$ . Given  $\epsilon_1 > 0$ , take  $\overline{L}_n$  as above such that  $|g_n - g| < \epsilon_1$ , then the model  $\mathcal{X}_n$  determines a rational polytopal decomposition  $(\mathcal{C}_n)_\Lambda$  of  $\mathbb{R}^d/\Lambda$  which comes from a  $\Lambda$ -periodic rational polytopal decomposition  $\mathcal{C}_n$  of  $\mathbb{R}^d$ . And the function  $g_n$  is rational polytopal and strictly polytopal convex with respect to  $\mathcal{C}_n$ . By Proposition 2.2, the set of irreducible components of  $\mathcal{X}_n$  is identified with the set of  $\Lambda$ -translation classes of vertices in  $\mathcal{C}_n$ . Hence if we denote  $Y_{\mathbf{v}}$  the irreducible component corresponding to  $\mathbf{v}$ , then  $Y_{\mathbf{v}} = Y_{\mathbf{v}'}$  if and only if  $\mathbf{v} = \mathbf{v}' + \boldsymbol{\lambda}$  for some  $\boldsymbol{\lambda} \in \Lambda$ . By (3.1), we have

$$\mu_n = \frac{1}{e_n^d} \sum_{\mathbf{v} \in \mathfrak{F}} \deg_{\overline{\mathcal{L}}_n}(Y_{\mathbf{v}}) \delta_{p(\mathbf{v})} \quad (4.3)$$

where we recall that  $\mathfrak{F} \subset \mathbb{R}^d$  is the fixed fundamental domain of  $\Lambda$  and  $p : \mathbb{R}^d \rightarrow \mathbb{R}^d/\Lambda$  is the projection. We recall a formula in [9, p.366 (36)] which is deduced from [7, p.112 Corollary], that

$$\deg_{\overline{\mathcal{L}}_n}(Y_{\mathbf{v}}) = d! \cdot \text{vol}(\widehat{\mathbf{v}}^{e_n g_n}).$$

Hence we have

$$(4.3) = d! \sum_{\mathbf{v} \in \mathfrak{F}} \text{vol}(\widehat{\mathbf{v}}^{g_n}) \delta_p(\mathbf{v}). \quad (4.4)$$

For any integer  $N > 0$ , we divide  $\mathfrak{F}$  into  $N^d$  blocks as follows. For  $(b_1, \dots, b_d) \in \{0, 1, \dots, N-1\}^d$ , let

$$\mathfrak{F}_{b_1, \dots, b_d}^{(N)} = \left\{ \mathbf{x} = x_1 \boldsymbol{\lambda}_1 + \dots + x_d \boldsymbol{\lambda}_d \mid \frac{b_i}{N} \leq x_i < \frac{b_i + 1}{N} \right\}.$$

Then  $\mathfrak{F} = \bigsqcup \mathfrak{F}_{b_1, \dots, b_d}^{(N)}$  and  $\overline{\mathfrak{F}_{b_1, \dots, b_d}^{(N)}}$  is a  $d$ -dimensional rational polytope. For any  $\epsilon_2 > 0$ , there exists  $N(\epsilon_2) > 0$  such that for any  $N \geq N(\epsilon_2)$ ,

$$\max_{\mathbf{x} \in \mathfrak{F}_{b_1, \dots, b_d}^{(N)}} g_{ij}(\mathbf{x}) - \min_{\mathbf{x} \in \mathfrak{F}_{b_1, \dots, b_d}^{(N)}} g_{ij}(\mathbf{x}) < \epsilon_2 \quad (4.5)$$

for all  $i, j = 1, \dots, d$  and  $(b_1, \dots, b_d)$ . We now assume  $N$  is that large and consider on a block, say without loss of generality,  $\mathfrak{F}^{(N)} = \mathfrak{F}_{0, \dots, 0}^{(N)}$ . Then

$$\mu_n|_{\mathfrak{F}^{(N)}}(f) = d! \sum_{\mathbf{v} \in \mathfrak{F}^{(N)}} \text{vol}(\widehat{\mathbf{v}}^{g_n}) f(\mathbf{v}) \leq d! \cdot \text{vol}(\Delta^{(N)}) \cdot \sup_{\mathbf{x} \in \mathfrak{F}^{(N)}} f(\mathbf{x})$$

where  $\Delta^{(N)}$  is the convex hull of  $\mathbf{m}_\Delta$  (pegs induced by  $g_n$ ) for (finitely many)  $d$ -dimensional polytopes  $\Delta \in \mathcal{C}_n$  such that  $\Delta \cap \mathfrak{F}^{(N)} \neq \emptyset$ . Now we are going to give an upper bound for this volume. Let  $\mathbf{x}_0$  be the point  $\frac{1}{2N}(\boldsymbol{\lambda}_1 + \dots + \boldsymbol{\lambda}_d)$  which is the center of symmetry of  $\overline{\mathfrak{F}^{(N)}}$ . The volume  $\text{vol}(\Delta^{(N)})$  will keep unchanged under following operations:

a) We replace  $g_n$  by  $\tilde{g}_n$  where

$$\tilde{g}_n(\mathbf{x}) = g_n(\mathbf{x}) - \nabla g(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) - g(\mathbf{x}_0);$$

(we also let  $\tilde{g}(\mathbf{x}) = g(\mathbf{x}) - \nabla g(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) - g(\mathbf{x}_0)$ .)

b) We make a translation  $\mathbf{x}' = \mathbf{x} - \mathbf{x}_0$ ;

c) We apply a rotation  $\mathbf{x}'' = R \cdot \mathbf{x}'$  for some  $R \in \mathbf{SO}_d$ .

Hence we may assume that

a')  $\nabla g(\mathbf{0}) = \mathbf{0}$  and  $|g_n - g| < \epsilon_1$ ;

b')  $g(\mathbf{0}) = 0$ ;

c')

$$(g_{ij}(\mathbf{0})) = \begin{pmatrix} h_{11} & & & \\ & h_{22} & & \\ & & \ddots & \\ & & & h_{dd} \end{pmatrix}$$

with  $h_g < h_{11} \leq \dots \leq h_{dd} < H_g$ ;

d')

$$\mathfrak{F}^{(N)} = \left\{ \mathbf{x} = x_1 \boldsymbol{\lambda}'_1 + \cdots + x_d \boldsymbol{\lambda}'_d \mid -\frac{1}{2N} \leq x_i < \frac{1}{2N} \right\}$$

where  $\boldsymbol{\lambda}'_i = R \cdot \boldsymbol{\lambda}_i$  for certain  $R \in \mathbf{SO}_d$ .

For any  $\epsilon_3 \geq 0$ , we also introduce the following

$$\mathfrak{F}_{g,\epsilon_3}^{(N)} = \left\{ \mathbf{x}' = (1 + \epsilon_3)h_{11}x_1\mathbf{e}_1 + \cdots + (1 + \epsilon_3)h_{dd}x_d\mathbf{e}_d \mid \mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_d\mathbf{e}_d \in \mathfrak{F}^{(N)} \right\}.$$

The following lemma is obvious.

**Lemma 4.4.** *For any  $\mathbf{x}' \in \mathfrak{F}_{g,0}^{(N)}$ , the ball  $B\left(\mathbf{x}', \frac{\epsilon_3 h_g r_{\mathfrak{F}}}{N}\right)$  is contained in  $\mathfrak{F}_{g,\epsilon_3}^{(N)}$ .*

Now for any  $\mathbf{x} \in \mathfrak{F}^{(N)}$ , we have

$$g_i(\mathbf{x}) = \int_0^1 \nabla_{\mathbf{x}} \nabla_{\mathbf{e}_i} g(t\mathbf{x}) dt.$$

By (4.5), we have

$$|g_i(\mathbf{x}) - h_{ii}x_i| \leq \epsilon_2(|x_1| + \cdots + |x_d|) \leq \frac{\epsilon_2 \cdot d R_{\mathfrak{F}}}{2N}.$$

The point  $\mathbf{x}' = h_{11}x_1\mathbf{e}_1 + \cdots + h_{dd}x_d\mathbf{e}_d$  is in  $\mathfrak{F}_{g,0}^{(N)}$ . Let  $\mathbf{m}_{\Delta}(\mathbf{x})$  be (any) peg of  $\Delta$  containing  $\mathbf{x}$ , then we have, by Lemma 3.8,

$$\|\mathbf{m}_{\Delta}(\mathbf{x}) - \mathbf{x}'\| \leq \|\mathbf{m}_{\Delta}(\mathbf{x}) - \nabla g(\mathbf{x})\| + \|\nabla g(\mathbf{x}) - \mathbf{x}'\| \leq 2\sqrt{\epsilon_1 \cdot H_g} + \frac{\epsilon_2 \cdot d^{\frac{3}{2}} R_{\mathfrak{F}}}{2N}.$$

Hence by Lemma 4.4, if

$$2\epsilon_3 h_g r_{\mathfrak{F}} \geq \epsilon_2 \cdot d^{\frac{3}{2}} R_{\mathfrak{F}} + 4N\sqrt{\epsilon_1 \cdot H_g},$$

then  $\mathbf{m}_{\Delta}(\mathbf{x}) \in \mathfrak{F}_{g,\epsilon_3}^{(N)}$ . Since the later is convex, we have  $\text{vol}(\Delta^{(N)}) \leq \text{vol}(\mathfrak{F}_{g,\epsilon_3}^{(N)}) = (1 + \epsilon_3)^d h_{11} \cdots h_{dd} \cdot \text{vol}(\mathfrak{F}^{(N)})$ . Now we let  $\epsilon_3 = \sqrt[d]{1 + \delta} - 1$ ,  $\epsilon_2 = \epsilon_3 h_g r_{\mathfrak{F}} d^{-\frac{3}{2}} R_{\Lambda}^{-1}$ . Then for a fixed  $N \geq N(\epsilon_2)$ , when  $n$  is large enough and hence

$$\epsilon_1 \leq \frac{1}{H_g} \left( \frac{\epsilon_3 h_g r_{\mathfrak{F}}}{4N} \right)^2,$$

we have

$$\mu_n|_{\mathfrak{F}^{(N)}}(f) \leq (1 + \delta)d! \cdot h_{11} \cdots h_{dd} \cdot \sup_{\mathbf{x} \in \mathfrak{F}^{(N)}} f(\mathbf{x}) \leq (1 + \delta) \sup_{\mathbf{x} \in \mathfrak{F}^{(N)}} f \cdot \text{Hess}_g(\mathbf{x}) \cdot \text{vol}(\mathfrak{F}^{(N)}).$$

Summing over all  $(b_1, \dots, b_d)$ , we have

$$\mu_n(f) \leq (1 + \delta) \sum_{(b_1, \dots, b_d)} \sup_{\mathbf{x} \in \mathfrak{F}_{b_1, \dots, b_d}^{(N)}} f \cdot \text{Hess}_g(\mathbf{x}) \cdot \text{vol}(\mathfrak{F}_{b_1, \dots, b_d}^{(N)}).$$

Let  $n \rightarrow \infty$  and then  $N \rightarrow \infty$ , we have

$$\mu(f) \leq (1 + \delta) \int_{\mathfrak{F}} f(\mathbf{x}) \text{Hess}_g(\mathbf{x}) d\mathbf{x}$$

which confirms the claim (4.2). □

## 5 A Calabi-Yau theorem

In this section, we state and prove the non-archimedean analogue of the Calabi-Yau theorem for totally degenerate abelian variety  $A$ .

*Classical Calabi-Yau theorem review.* Let us have a quick review of the famous Calabi conjecture which is proved by Yau in complex geometry. For details, we refer to Yau's original paper [15] and also the book [10, Chapter 5] by Joyce. For simplicity, we just state it for the algebraic case. Hence let  $M$  be a connected compact complex manifold of dimension  $d \geq 1$  and  $L$  an ample line bundle on it. Given any smooth metric  $\|\cdot\|$  on  $L$ , we have the Chern class  $\omega = c_1(L, \|\cdot\|)$  which is a (smooth)  $(1, 1)$ -form on  $M$ . It determines a measure, i.e., a top form  $\mu = \omega^{\wedge d}$  on  $M$ . We say  $\|\cdot\|$  is positive if  $\omega$  is positive definite everywhere. Then the measure  $\mu$  is obviously positive. The Calabi conjecture asserts that given any smooth positive  $(d, d)$ -form  $\mu'$  such that  $\int_M \mu' = \int_M \mu$ , there exists a smooth positive measure  $\|\cdot\|'$  on  $L$ , unique up to a scalar, such that  $\mu' = (\omega')^{\wedge d}$  where  $\omega' = c_1(L, \|\cdot\|')$ .

If we write  $\mu' = e^f \mu$  for a unique smooth real function  $f$  on  $M$ , then the Calabi conjecture asserts that there exists a unique smooth real function  $\phi$  such that

- (1)  $\omega + dd^c \phi$  is a positive  $(1, 1)$ -form;
- (2)  $\int_M \phi \mu = 0$ ;
- (3)  $(\omega + dd^c \phi)^{\wedge d} = e^f \mu$ .

If we choose a local coordinates  $z_1, \dots, z_d$  on an open set  $U$  in  $M$ , then  $(g_{\alpha\bar{\beta}})_{\alpha, \bar{\beta}=1, \dots, d}$  is an  $d \times d$  hermitian matrix, where  $g$  is the Riemannian metric associate with  $\omega$ . Then the condition (3) reads

(3')

$$\det \left( g_{\alpha\bar{\beta}} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta} \right) = e^f \det (g_{\alpha\bar{\beta}}) \quad (5.1)$$

which is a complex Monge-Ampère equation.

More generally,  $\omega$  could just be a Kähler form. Then the existence part of the following theorem is due to Yau and the uniqueness part is due to Calabi.

**Theorem 5.1.** *Let  $M, \omega$  be as above, then*

*(Existence, cf. [15, §4, Theorem 1]) For any  $f \in \mathcal{C}^k(M)$  ( $k \geq 3$ ), there exists  $\phi \in \mathcal{C}^{k+1, \alpha}(M)$  for any  $\alpha \in [0, 1)$  satisfying (1)-(3);*

*(Uniqueness, cf. [4], [15, §5, Theorem 3]) For any  $f \in \mathcal{C}^1(M)$ , there is at most one  $\phi \in \mathcal{C}^3(M)$  satisfying (1)-(3).*

*A non-archimedean analogue.* Recall that we have a totally degenerate abelian variety  $A$  of dimension  $d$  over  $k$  and an ample line bundle  $L$  on it. For any integrable metrized line bundle  $\bar{L} = (L, \|\cdot\|)$ , we define the measure  $c_1(\bar{L})^{\wedge d}$  on the analytic space  $A^{\text{an}}$ . Also, we have a skeleton  $i_A : \mathbb{R}^d / \Lambda \hookrightarrow A^{\text{an}}$ . The following is a Calabi-Yau theorem in the current setting for positive measures supported on this skeleton, which has certain smoothness in the real-analytic sense.



**Theorem 5.2 (Non-archimedean Calabi-Yau).** *Let  $A, L, i_A$ , be as above. For any  $\mu \in \mathcal{M}_{\geq 0}^k(\mathbb{R}^d/\Lambda)$  ( $k \geq 3$ ), there is a semi-positive metric  $\|\cdot\|$  on  $L$ , unique up to scalar, such that  $c_1(\overline{L})^{\wedge d} = (i_A)_*\mu$  where  $\overline{L} = (L, \|\cdot\|)$ . Moreover,  $\overline{L}$  is toric in the sense of Definition 3.3 whose corresponding Green function  $g$  is in  $\mathcal{G}_+(L) \cap \mathcal{C}^{k+1,\alpha}(\mathbb{R}^d)$  for any  $\alpha \in [0, 1)$ .*

*Proof.* The uniqueness part follows from the general theorem on the uniqueness [16, Theorem 1.1.1] proved by Yuan and Zhang.

Now we prove the existence. Recall that we have a canonical Green function  $g_{\text{can}}$  for  $L$  which determines a measure  $\mu_{\text{can}}$  on  $\mathbb{R}^d/\Lambda$  (which is just  $H_q$  times the Lebesgue measure). By Theorem 4.3, we only need to prove that for a given  $f \in \mathcal{C}^k(\mathbb{R}^d/\Lambda)$  ( $k \geq 3$ ) such that  $\int_{\mathbb{R}^d/\Lambda} e^f d\mathbf{x} = 1$ , there exists a function  $\phi \in \mathcal{C}^{k+1,\alpha}(\mathbb{R}^d/\Lambda)$  for any  $\alpha \in [0, 1)$  such that:

- The matrix  $((g_{\text{can}})_{ij} + \phi_{ij})_{i,j=1,\dots,d}$  is positive definite;
- It satisfies the real Monge-Ampère equation

$$\det \left( (g_{\text{can}})_{ij} + \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \frac{H_q}{d!} e^f; \quad (5.2)$$

- If  $f \in \mathcal{C}^\infty(\mathbb{R}^d/\Lambda)$ , then  $\phi \in \mathcal{C}^\infty(\mathbb{R}^d/\Lambda)$ .

We would like to deduce it from the complex case, i.e., theorem 5.1. We introduce the following manifold

$$\mathbb{A} = \mathbb{R}^d/\Lambda \oplus \mathbb{R}^d/\Lambda$$

where we write  $(x_1, \dots, x_d; y_1, \dots, y_d)$  for the usual chart. The tangent bundle has a canonical splitting  $\mathcal{T}_{\mathbb{A}} = \mathcal{T}_1 \oplus \mathcal{T}_2$  where  $\mathcal{T}_i$  is the pull-back of the tangent bundle on the  $i$ -th  $\mathbb{R}^d/\Lambda$ . Write  $u_i = \partial x_i$  and  $v_i = \partial y_i$  and define a complex structure  $J$  on  $\mathcal{T}_{\mathbb{A}}$  by  $Ju_i = v_i$ ,  $Jv_i = -u_i$  ( $i = 1, \dots, d$ ). Then as a complex manifold,  $\mathbb{A}$  is isomorphic to  $\mathbb{C}^d/\Lambda \oplus \Lambda$ .

We define

$$g((u_i, v_j), (u_{i'}, v_{j'})) = \frac{1}{2} ((g_{\text{can}})_{ii'} + (g_{\text{can}})_{jj'}).$$

Then  $g$  is a Riemannian metric on  $\mathbb{A}$  and  $\omega(w, w') = g(Jw, w')$  is a Kähler metric on  $(\mathbb{A}, J)$  with  $\mu = \omega^{\wedge d}$ . Define  $\mathbb{f}(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})$  which is in  $\mathcal{C}^k(\mathbb{A})$ . Then  $((\mathbb{A}, J), \omega, \mathbb{f})$  is in the situation of Theorem 5.1. Applying this theorem, we see that there is a unique function  $\Phi \in \mathcal{C}^{k+1,\alpha}(\mathbb{A})$  for any  $\alpha \in [0, 1)$  satisfying (1)-(3). If we write the Monge-Ampère equation (5.1) explicitly in the current situation, we see that  $\Phi$  satisfies

$$\det \left( (g_{\text{can}})_{\alpha\bar{\beta}} + \frac{\partial^2 \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{z}_\alpha \partial \bar{\mathbf{z}}_\beta} \right) = \frac{H_q}{d!} e^{\mathbb{f}(\mathbf{x})} \quad (5.3)$$

where  $\mathbf{z}_\alpha = \mathbf{x}_\alpha + i\mathbf{y}_\alpha$  and  $\bar{\mathbf{z}}_\beta = \mathbf{x}_\beta - i\mathbf{y}_\beta$ . For any  $\mathbf{y}_0 \in \mathbb{R}^d/\Lambda$ , let  $\Phi_{\mathbf{y}_0}(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y} - \mathbf{y}_0)$ . Then  $\Phi_{\mathbf{y}_0}$  is also a  $\mathcal{C}^{5,\alpha}$ -solution satisfying (1)-(3). Hence by the uniqueness,  $\Phi_{\mathbf{y}_0} = \Phi$  for any  $\mathbf{y}_0$ , i.e.,

$$\frac{\partial \Phi}{\partial \mathbf{y}_i} \equiv 0; \quad i = 1, \dots, d.$$

Restricting to  $\mathbb{R}^d/\Lambda \times \{\mathbf{0}\}$ , we see that  $\phi(\mathbf{x}) := \Phi(\mathbf{x}, \mathbf{0}) \in \mathcal{C}^{k+1,\alpha}(\mathbb{R}^d/\Lambda)$  for any  $\alpha \in [0, 1)$ , satisfies the real Monge-Ampère equation (5.2) and such that  $((g_{\text{can}})_{ij} + \phi_{ij})$  is positive definite. Hence the theorem is proved.  $\square$

*Remark 5.3.* The restriction on the field  $k$  is not necessary. In fact, all results and argument remain valid for any algebraically closed non-archimedean field whose valuation is non-trivial. One only need to use the generalized definition of Chambert-Loir's measure given by Gubler in [9, §3].

## References

- [1] Berkovich, V.G.: *Spectral theory and analytic geometry over non-archimedean fields*. Math. Surv. Monogr., vol.33. Amer. Math. Soc., Providence (1990)
- [2] Berkovich, V.G.: Smooth  $p$ -adic analytic spaces are locally contractible. *Invent. Math.* **137**, 1-84 (1999)
- [3] Bosch, S., Lütkebohmert, W.: Degenerating abelian varieties. *Topology* **30**(4), 653-698 (1991)
- [4] Calabi, E.: The space of Kähler metrics. In *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954*, vol. 2, 206-207. North-Holland, Amsterdam (1956)
- [5] Chambert-Loir, A.: Mesure et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.* **595**, 215-235 (2006)
- [6] Fresnel, J., van der Put, M.: *Rigid analytic geometry and its applications*. Prog. Math., vol. 218. Birkhäuser, Boston, MA (2004)
- [7] Fulton, W.: *Introduction to toric varieties*. Ann. Math. Stud., vol. 131. Princeton University Press, Princeton, NJ (1993)
- [8] Gubler, W.: Local heights of subvarieties over non-archimedean fields. *J. Reine Angew. Math.* **498**, 61-113 (1998)
- [9] Gubler, W.: Tropical varieties for non-archimedean analytic spaces. *Invent. Math.* **169**, 321-376 (2007)
- [10] Joyce, D. D.: *Compact manifolds with special holonomy*. Oxford Mathematics Monographs. Oxford University Press, Oxford (2000)
- [11] McMullen, P.: Duality, sections and projections of certain Euclidean tilings. *Geom. Dedicata* **49**(2), 183-202 (1994)
- [12] Mumford, D.: *Abelian varieties*. Tata Institute of Fundamental Research. Studies in mathematics, 5. Oxford University Press, Oxford (1970)
- [13] Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings. *Compos. Math.* **24**, 239-272 (1972)
- [14] Yau, S.-T.: On Calabi's conjecture and some new results in algebraic geometry. In *Proceedings of the National Academy of Sciences of the U.S.A.*, 74, 1798-1799 (1977)
- [15] Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Commun. Pure Appl. Math.* **31**, 339-411 (1978)
- [16] Yuan, X., Zhang, S.-W.: Calabi theorem and algebraic dynamics. preprint. available at <http://www.math.columbia.edu/~szhang/papers/Preprints.htm> (2009)
- [17] Zhang, S.-W.: Positive line bundles on arithmetic varieties. *J. Am. Math. Soc.* **8**, 187-221 (1995)
- [18] Zhang, S.-W.: Small points and adelic metrics. *J. Alg. Geom.* **4**, 281-300 (1995)
- [19] Zhang, S.-W.: Gross-Schoen cycles and dualizing sheaves. *Invent. Math.* **179**, 1-73 (2010)